Dual quasitriangular structures related to the Temperley–Lieb algebra

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Abstract

We consider nonquasiclassical solutions to the quantum Yang–Baxter equation and the corresponding quantum cogroups $\mathrm{Fun}(SL(S))$ constructed earlier in [G]. We give a criterion of the existence of a dual quasitriangular structure in the algebra $\mathrm{Fun}(SL(S))$ and describe a large class of such objects related to the Temperley–Lieb algebra satisfying this criterion. We show also that this dual quasitriangular structure is in some sense nondegenerate.

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1 Introduction

It became clear after the works of one of the authors [G] that besides the well-known deformational (or quasiclassical) solutions to the quantum Yang–Baxter equation (QYBE) there exists a lot of other solutions that differ drastically from the former ones. Let us explain this in more detail. Let $\mathbf{V}$

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be a linear space over the field $K = \mathbf{C}$ or R. We call a *Yang–Baxter operator* a solution $S : \mathbf{V}^{\otimes 2} \to \mathbf{V}^{\otimes 2}$ to the QYBE

$$S^{12} S^{23} S^{12} = S^{23} S^{12} S^{23}, \ S^{12} = S \otimes \mathrm{id}, \ S^{23} = \mathrm{id} \otimes S.$$

A Yang–Baxter operator satisfying a second degree equation

$$(\mathrm{id} + S)(q\,\mathrm{id} - S) = 0$$

was called in [G] a *Hecke symmetry*. The quantum parameter $q \in K$ is assumed to be generic.

It is natural to associate to a Hecke symmetry two algebras defined as follows

$$\wedge_+(\mathbf{V}) = T(\mathbf{V})/\{\mathrm{Im}(q\,\mathrm{id} - S)\}, \ \wedge_-(\mathbf{V}) = T(\mathbf{V})/\{\mathrm{Im}(\mathrm{id} + S)\}.$$

They are q-counterparts of the symmetric and skew-symmetric algebras of the space $\mathbf{V}$, respectively.

Let us denote $\wedge_\pm^l(\mathbf{V})$ the degree l homogeneous component of the algebra $\wedge_\pm(\mathbf{V})$. It was shown in [G] that the *Poincaré series*

$$P_\pm(t) = \sum \dim \wedge_\pm^l (\mathbf{V})t^l$$

of the algebras $\wedge_\pm(\mathbf{V})$ for a generic q satisfy the standard relation

$$P_+(t)P_-(-t) = 1.$$

Moreover, if the series $P_-(t)$ is a polynomial with leading coefficient 1, it is reciprocal. Hecke symmetries of such type and the corresponding linear spaces $\mathbf{V}$ are called *even*.

A particular case of an even Hecke symmetry is provided by the quantum groups $U_q(sl(n))$: the operator $S = \sigma \rho^{\otimes 2}(\mathcal{R})$ (where $\mathcal{R}$ is the corresponding universal R-matrix, σ is the flip and $\rho : sl(n) \to \mathrm{End}(\mathbf{V})$ is the fundamental vector representation) is just such a type of solution to the QYBE. In fact, we have a family of operators S_q and we recover the standard flip σ for $q = 1$. Namely, in this sense we call such Yang–Baxter operators (and all related objects) *deformational* or *quasiclassical*.

The Poincaré series for Hecke symmetries of such type coincide with the classical ones. Thus, in this case we have $P_-(t) = (1 + t)^n$ with $n = \dim \mathbf{V}$ and, therefore, $P_+(t) = (1 - t)^{-n}$.

However, this is no longer true in general case. As shown in [G], for any $n = \dim \mathbf{V}$ and any integer p, $2 \leq p \leq n$ there exists a nonempty family of nontrivial even Hecke symmetries such that $\deg P_-(t) = p$. The integer p is called the *rank* of the even Hecke symmetry S or of the corresponding space $\mathbf{V}$.

The classification problem of all even Hecke symmetries of a given rank p is still open. However, all such symmetries of rank $p = 2$ are completely classified.

Let us observe that the case $p = 2$ is related to the Temperley–Lieb (TL) algebra, since the projectors $P_-^i : \mathbf{V}^{\otimes m} \to \mathbf{V}^{\otimes m}$, $1 \leq i \leq m - 1$ defined by

$$P_-^i = \frac{(q\,\mathrm{id} - S^{i,i+1})}{(q+1)}$$

where $S^{i,i+1}$ is the operator S acting onto the i-th and $(i+1)$-th components of $\mathbf{V}^{\otimes m}$, generate a TL algebra. Let us recall that a TL algebra is the algebra generated by t_i, $1 \leq i \leq n - 1$ with the following relations:

$$t_i^2 = t_i, \ t_i\, t_{i\pm 1}\, t_i = \lambda t_i \ , t_i\, t_j = t_j\, t_i \ \ |i - j| > 1.$$

(In the case under consideration $\lambda = q(1 + q)^{-2}$.) In what follows even rank 2 Hecke symmetries will be called symmetries of *TL type*.

It is possible to assign to any YB operator S the famous "RTT=TTR" algebra. In the sequel it is called the *quantum matrix algebra* and denoted by $\mathbf{A}(S)$. If S is an even Hecke symmetry of the TL type, in this algebra there exists a so-called *quantum determinant* (it was introduced in [G]). If it is a central element, it is possible to define a *quantum cogroup* $\mathrm{Fun}(SL(S))$ looking like the famous quantum function algebra $\mathrm{Fun}_q(SL(n))$ (in [G] this algebra was called a quantum group)[1]. Let us note that these quantum cogroups $\mathrm{Fun}(SL(S))$ possess Hopf algebra structures.

However, until now no corepresentation theory of such nonquasiclassical quantum cogroups has been constructed. In particular, it is not clear whether any finite-dimensional $\mathrm{Fun}(SL(S))$-comodule is semisimple for a generic q. Nevertheless this problem seems to be of great interest, since the nonquasi-classical solutions to the QYBE provides us a new type symmetries, which

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[1]Let us note that a subclass of objects of such type was independently introduced in [DL].

differ drastically from classical or supersymmetries. (The simplest models possessing such a symmetry of new type corresponding to an involutive S, namely, a "nonquasiclassical harmonic oscillator" was considered in [GRZ]. Let us observe that the partition functions of such models can be expressed in terms of the Poincaré series corresponding to the initial symmetry.)

The present paper is the first in a series aimed at a better understanding the structure of such nonquasiclassical symmetries. More precisely, we discuss here two problems: first, whether the quantum cogroups $\operatorname{Fun}(SL(S))$ have quasitriangular structure and second, what is an explicit description of their dual objects?

It is well known that the notion of a quasitriangular structure was introduced by V. Drinfeld. In fact this notion was motivated by the quantum groups $U_q(g)$. These objects have an explicit description due to V. Drinfeld and M. Jimbo in terms of deformed Cartan–Weyl system $\{H_\alpha, X_{\pm\alpha}\}$ (cf., i.e., [CP]). Thus this construction allows one to develop a representation theory of quantum groups.

In the nonquasiclassical case under consideration such a description does not exist. And the problem of an appropriate description of objects dual to the quantum cogroups $\operatorname{Fun}(SL(S))$ is of great interest. (The duality in the present paper is understood in the algebraic sense, i.e., all dual objects are *restricted*).

We attack this problem here by means of the so-called *canonical pairing*. Such a pairing can be defined on any algebra $\mathbf{A}(S)$ (an algebra $\mathbf{A}(S)$ equipped with such a pairing is called *dual quasitriangular*). Nevertheless, only under some additional conditions this pairing can be descended to the quantum cogroup $\operatorname{Fun}(SL(S))$. We show here that this condition is satisfied for the quantum cogroup $\operatorname{Fun}(SL(S))$ related to TL algebras.

Moreover, we show that in this case the canonical pairing is nondegenerate when restricted to the span of the generators of this algebra. This is the main difference between the quasiclassical and nonquasiclassical cases: in the former case this pairing is degenerate. This is a reason why we cannot introduce an object dual to $\operatorname{Fun}_q(SL(n))$ by means of this pairing. Finally, following the paper [RTF], in this case we must introduce an additional pairing (associated in a similar way to the YB operator S^{-1}). And the above mentioned deformed Cartan–Weyl basis in $U_q(g)$ can be constructed by means of both pairings (for this construction the reader is referred to [RTF], cf. also Remark 1).

As for the nonquasiclassical case we can equip the basic space **V** (using nondegeneracy of the canonical pairing in the mentioned sense) with a structure of a Fun($SL(S)$)-module. Moreover, we can equip any tensor power of the space **V** with such a structure. Finally, we get a new tool to study tensor categories generated by such spaces.

The paper is organized as follows. In the next Section we introduce the notion of a dual quasitriangular structure and describe the one connected with the quantum matrix algebras in terms of the canonical pairing. In Section 3 we give the condition ensuring the existence of such a structure on the algebra Fun($SL(S)$) mentioned above and in Section 4 we show that this condition is satisfied for a large family of such algebras related to the TL algebras. We conclude the paper with a proof of the nondegeneracy of the canonical pairing (in the above sense) for algebras from this family (Section 5) and with a discussion of a hypothetical representation theory for the algebra Fun($SL(S)$) (Section 6).

2   Dual quasitriangular structure

The notion of a dual quasitriangular bialgebra (in particular, a Hopf algebra) was introduced by Sh. Majid (see [M] and the references therein) as a dualization of the notion of a quasitriangular bialgebra due to V. Drinfeld. By definition, a dual quasitriangular bialgebra is a bialgebra equipped with some pairing similar to the one defined on the cogroups $\text{Fun}_q(G)$ by means of the quantum universal R-matrix $\mathcal{R}$:

$$a \otimes b \rightarrow \langle\langle a, b \rangle\rangle = \langle a \otimes b, \mathcal{R} \rangle \; a, \, b \in \text{Fun}_q(G).$$

Here the pairing $\langle \, , \, \rangle$ is that between $\text{Fun}_q(G)$ and $U_q(g)$ extended to their tensor powers.

More precisely, one says that a bialgebra (or a Hopf algebra) $\mathcal{A}$ is equipped with a *dual quasitriangular structure* and it is called a *dual quasitriangular bialgebra* (or *Hopf algebra*), if it is endowed with a pairing

$$\langle\langle \, , \, \rangle\rangle : \mathcal{A}^{\otimes 2} \rightarrow K$$

satisfying the following axioms

$$(\text{i}) \qquad \langle\langle a, \, bc \rangle\rangle = \langle\langle a_{(1)}, \, c \rangle\rangle \langle\langle a_{(2)}, \, b \rangle\rangle,$$

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(ii) $\qquad \langle\langle ab,\, c\rangle\rangle = \langle\langle a,\, c_{(1)}\rangle\rangle\langle\langle b,\, c_{(2)}\rangle\rangle,$

(iii) $\qquad \langle\langle a_{(1)},\, b_{(1)}\rangle\rangle a_{(2)}b_{(2)} = b_{(1)}a_{(1)}\langle\langle a_{(2)},\, b_{(2)}\rangle\rangle,$

(iv) $\qquad \langle\langle 1,\, a\rangle\rangle = \varepsilon(a) = \langle\langle a,\, 1\rangle\rangle$

for all $a, b, c \in \mathcal{A}$, where $\varepsilon : \mathcal{A} \to K$ is the counit of $\mathcal{A}$ and $\Delta : \mathcal{A} \to \mathcal{A}^{\otimes 2}$, $\Delta(a) = a_{(1)} \otimes a_{(2)}$ is the coproduct. If $\mathcal{A}$ is a Hopf algebra and $\gamma : \mathcal{A} \to \mathcal{A}$ is its antipode, we impose a complementary axiom

$$\text{(v)}\quad \langle\langle a,\, b\rangle\rangle = \langle\langle \gamma(a),\, \gamma(b)\rangle\rangle.$$

(If the pairing is invertible in sense of [M], p. 48 the axioms (i)–(iii) imply those (iv) and (v), cf [M].)

Let us note that the axioms (i), (ii), (iv), (v) mean that the product (resp., the coproduct, the unit, the counit, the antipode) of the algebra $\mathcal{A}$ is dual to the coproduct (resp. the product, the counit, the unit, the antipode) of the algebra $\mathcal{A}^{op}$ where $\mathcal{A}^{op}$ denotes as usually the bialgebra $\mathcal{A}$ whose product is replaced by the opposite one. So, in fact, we have the pairing of bialgebras (Hopf algebras)

$$\langle\langle\ ,\ \rangle\rangle : \mathcal{A} \otimes \mathcal{A}^{op} \to K. \tag{1}$$

In some sense the notion of a dual quantum bialgebra is more fundamental than that of quasitriangular one for the following reason. It is well known that the most popular construction of a quasitriangular Hopf algebra is given by the famous Drinfeld–Jimbo quantum group $U_q(g)$. Usually it is introduced by means of the Cartan–Weyl generators $\{H_\alpha, X_{\pm\alpha}\}$ and certain relations between them which are quantum (or q-) analogues of the ordinary ones. However, this approach is valid only in the quasiclassical case.

In the general case, including the nonquasiclassical objects, we should first introduce dual quasitriangular bialgebras (or Hopf algebras) and only after that we can proceed to introduce their dual objects. Moreover, an explicit description of the latter objects depends on the properties of the canonical pairing and they are not similar in quasiclassical and nonquasiclassical cases.

Let us describe now a regular way to introduce the dual quasitriangular bialgebras (and Hopf algebras) associated to the YB operators discussed above. Let $\mathbf{V}$ be a linear space equipped with a nontrivial Yang–Baxter operator $S : \mathbf{V}^{\otimes 2} \to \mathbf{V}^{\otimes 2}$. Let us fix a basis $\{x_i\}$ in $\mathbf{V}$ and denote by S_{ij}^{kl} the entries of the operator S $(S(x_i \otimes x_j) = S_{ij}^{kl} x_k \otimes x_l)$. From here on summation on repeated indices is assumed.

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Let us consider a matrix t with entries t_k^l, $1 \le k, l \le n = \dim \mathbf{V}$. The bialgebra $\mathbf{A}(S)$ of *quantum matrices* associated to S is defined as the algebra generated by 1 and n^2 indeterminates $\{t_k^l\}$, satisfying the following relations

$$S(t \otimes t) = (t \otimes t)S, \text{ or in a basis form } S_{ij}^{mn} t_m^p t_n^q = t_i^u t_j^v S_{uv}^{pq}.$$

This algebra possesses a bialgebra structure, being equipped with the co-matrix coproduct $\Delta(1) = 1 \otimes 1$, $\Delta(t_i^j) = t_i^p \otimes t_p^j$ and the counit $\varepsilon(1) = 1$, $\varepsilon(t_i^j) = \delta_i^j$.

This is just the famous "RTT=TTR" bialgebra introduced in [RTF]. Let us fix $c \in K$, $c \ne 0$, and equip this algebra with a dual quasitriangular structure by setting

$$\langle\langle 1, t_i^k \rangle\rangle_c = \delta_i^k = \langle\langle t_i^k, 1 \rangle\rangle_c \text{ and } \langle\langle t_i^k, t_j^l \rangle\rangle_c = c\, S_{ji}^{kl}$$

and extending the pairing to the whole $\mathbf{A}(S)^{\otimes 2}$ by using the above axioms (i), (ii) and (iv).

We leave to the reader to check that this extension is well defined (here it is precisely the QYBE that plays the crucial role) and, moreover, the axiom (iii) is satisfied as well.

The pairing $\langle\langle\ ,\ \rangle\rangle_c$ will be called *canonical*.

Let us remark that such a canonical pairing is usually considered with $c = 1$. However, we need this complementary "degree of freedom" to make the pairing $\langle\langle\ ,\ \rangle\rangle_c$ compatible with the equation $\det t = 1$ (cf. below). Another (but equivalent) way consists in replacing the Hecke symmetry S by cS. We drop the index c if $c = 1$.

Thus, the bialgebra $\mathbf{A}(S)$ can be *canonically* equipped with a dual quasitriangular structure.

Nevertheless, this bialgebra does not possess any Hopf algebra structure since no antipode is defined in it. To get a Hopf algebra, we must either impose the complementary equation $\det t = 1$, i.e., pass to the quotient of the algebra $\mathbf{A}(S)$ by the ideal generated by the element $\det t - 1$ (assuming the determinant $\det t$ to be well defined)[2] or add to the algebra a new generator $\det^{-1}$. In the latter case we obtain Hopf algebras (quantum cogroup) of GL type (cf. [G]).

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[2]In the sequel we will restrict ourselves to quantum cogroups of SL type whose construction was suggested in [G]. Quantum cogroups of SO or Sp type and the corresponding dual quasitriangular structures will be discussed elsewhere.

In any case it is necessary to check that the above dual quasitriangular structure on the algebra $\mathbf{A}(S)$ can be transferred to the final quantum cogroup. As for standard quantum function algebras $\mathrm{Fun}_q(G)$ dual to the quantum groups $U_q(g)$ (for a classical simple Lie algebra g) this follows automatically by duality. However, in general this is no longer true. In Section 3 we will give a necessary and sufficient condition ensuring the existence of a dual quasitriangular structure on a SL type quantum cogroup.

Thus, it is possible to associate a dual quasitriangular bialgebra to any YB operator and a dual quasitriangular Hopf algebra (of SL type) to some of them. These algebras look like the function algebras $\mathrm{Fun}(G)$ on a ordinary (semi) group G. This means that we can equip the space $\mathbf{V}$ with a (right to be concrete) comodule structure over the algebra $\mathrm{Fun}(SL(S))$ by

$$\Delta : \mathbf{V} \to \mathbf{V} \otimes \mathbf{A}(S), \ \ \Delta(x_i) = x_j \otimes t_i^j.$$

Therefore any tensor power of the space $\mathbf{V}$ also becomes a right $\mathbf{A}(S)$-comodule.

However, these powers are not in general irreducible as comodules over the above coalgebra. Unfortunately, up to now no corepresentation theory of quantum cogroups in nonquasiclassical cases has been constructed yet (such a hypothetical theory in the case connected to the TL algebra is discussed in Section 6). It is worth saying that even in the quasiclassical case it is possible to use quantum cogroups $\mathrm{Fun}_q(G)$ instead of the quantum groups $U_q(g)$ themselves. However, technically it is more convenient to work with the latter objects.

In the nonquasiclassical case an interesting problem arises: what is an appropriate description of the objects dual to the bialgebras $\mathbf{A}(S)$ or of their quotient of the SL type. If the canonical pairing (1) is nondegenerate, we can consider the bialgebra $\mathbf{A}(S)^{op}$ as the dual object to that $\mathbf{A}(S)$ (and similarly for their quotients of the SL type).

This is just (conjecturally) the case of the Hecke symmetries of TL type. We show that (at least for a large family of such symmetries) the canonical pairing, being restricted to the space $\mathbf{T} = \mathrm{Span}(t_i^j)$, is nondegenerate for a generic q. Nevertheless, this weak version of nondegeneracy is sufficient to equip the initial space $\mathbf{V}$ with a structure of a left $\mathbf{A}(S)^{op}$-module (and therefore, with that of a right $\mathbf{A}(S)$-module).

Having in mind the usual procedure (cf., i.e., [M]) we put

$$t_i^j \rhd x_k = x_m \langle\langle t_k^m, t_i^j \rangle\rangle_c = c S_{ik}^{mj} x_m,$$

where $t_i^j \rhd x_k$ denotes the result of applying the element $t_i^j \in \mathbf{A}(S)^{op}$ to $x_k \in \mathbf{V}$.

Remark 1 Let us observe that if the canonical pairing is degenerate, then the above action is still well defined, but $\mathbf{V}$ becomes reducible as an $\mathbf{A}(S)^{op}$-module since it contains the $\mathbf{A}(S)^{op}$-module $\mathrm{Im}(\rhd)$, where $\rhd : \mathbf{A}(S)^{op} \otimes \mathbf{V} \to \mathbf{V}$ is the above map and this module is a proper submodule in $\mathbf{V}$.

It is just the case related to the quantum groups $U_q(g)$. This is a reasons why one needs a complementary pairing. More precisely, let us introduce (following [RTF]) two sets of generators $(L^+)_i^j$ and $(L^-)_i^j$ and define the pairing between the spaces $\mathbf{L}^+ = \mathrm{Span}((L^+)_i^j)$, $\mathbf{L}^- = \mathrm{Span}((L^-)_i^j)$ and $\mathbf{T} = \mathrm{Span}(t_i^j)$ as follows

$$\langle\langle t_i^k, (L^+)_j^l \rangle\rangle = S_{ji}^{kl}, \ \langle\langle t_i^k, (L^-)_j^l \rangle\rangle = (S^{-1})_{ji}^{kl}.$$

In fact, in this way we have introduced a pairing between the spaces $\mathbf{L}^+ \oplus \mathbf{L}^-$ and $\mathbf{T}$. Of course, this pairing is degenerate on $\mathbf{L}^+ \oplus \mathbf{L}^-$ but it becomes nondegenerate on $\mathbf{T}$. There exists a natural way to extend the above pairing up to that $(\mathbf{L}^+ \oplus \mathbf{L}^-) \otimes \mathbf{A}(S) \to K$, cf. [RTF]. Thus, the space $\mathbf{L}^+ \oplus \mathbf{L}^-$ is embedded into the algebra $\mathbf{A}(S)^*$ dual to $\mathbf{A}(S)$. The subalgebra of $\mathbf{A}(S)^*$ generated by 1 and the space $\mathbf{L}^+ \oplus \mathbf{L}^-$ is called in [RTF] the *algebra of regular functions* on $\mathbf{A}(S)$. (Moreover, in [RTF] the elements $(L^+)_i^j$ are expressed in terms of the generators of the quantum groups $U_q(g)$.) In a similar way we can define such an algebra in the nonquasiclassical case under consideration, but since the canonical pairing is nondegenerate for a generic q, we restrict ourselves to the generators $(L^+)_i^j$.

3 Dual quasitriangular algebras related to the even Hecke symmetries

First, we recall some facts about the cogroups $\mathrm{Fun}(SL(S))$ introduced in [G].

Let us fix an even Hecke symmetry $S : \mathbf{V}^{\otimes 2} \to \mathbf{V}^{\otimes 2}$ of rank $p \geq 2$. Let us denote by $P_-^{(p)}$ the projector of $\mathbf{V}^{\otimes p}$ onto its skew-symmetric component

$\wedge_{-}^{p}(\mathbf{V})$ (an explicit form of this projector is given in [G]). Then by definition $\dim \operatorname{Im} P_{-}^{(p)} = 1$ and (assuming a base $\{x_i\} \in \mathbf{V}^{\otimes p}$ to be fixed)

$$P_{-}^{(p)} x_{i_1} x_{i_2} \ldots x_{i_p} = u_{i_1 i_2 \ldots i_p} v^{j_1 j_2 \ldots j_p} x_{j_1} x_{j_2} \ldots x_{j_p}$$

with $u_{i_1 i_2 \ldots i_p} v^{i_1 i_2 \ldots i_p} = 1$ (hereafter we drop the sign $\otimes$).

The tensors $U = (u_{i_1 i_2 \ldots i_p})$ and $V = (v^{j_1 j_2 \ldots j_p})$ are quantum analogue of the Levi–Civita ones.

Let us consider the bialgebra $\mathbf{A}(S)$ corresponding to the given even Hecke symmetry S and introduce a distinguished element in it

$$\det t = u_{i_1 i_2 \ldots i_p} t^{i_1}_{j_1} \ldots t^{i_p}_{j_p} v^{j_1 j_2 \ldots j_p}.$$

In [G] it was shown that this element is group-like, i.e.,

$$\Delta \left(\det t \right) = \det t \otimes \det t.$$

It was called a *quantum determinant.*

Under the additional condition that this determinant is central (in general this is not so), we introduce an analogue $\operatorname{Fun}(SL(S))$ of the quantum functional algebra $\operatorname{Fun}_q(SL(n))$ as the quotient algebra of $\mathbf{A}(S)$ over the ideal generated by the element $\det t - 1$. This quotient inherits a bialgebra structure but, moreover, it possesses a Hopf structure (for an explicit description of the antipode, the reader is referred to [G]). Our intermediate aim is to study whether it is possible to equip the algebra $\operatorname{Fun}(SL(S))$ with a dual quasitriangular structure?

It is evident that the dual quasitriangular structure on $\mathbf{A}(S)$ defined above can be descended to $\operatorname{Fun}(SL(S))$ iff

$$\langle\langle \det t,\, a \rangle\rangle_c = \varepsilon(a) = \langle\langle a,\, \det t \rangle\rangle_c \tag{2}$$

for any $a \in \mathbf{A}(S)$. Using the fact that the quantum determinant is a group-like element, it is possible to show that these relations are valid for any a if they are true for $a = t^j_i$. (As for $a = 1$, relation (2) follows immediately from $u_{i_1 i_2 \ldots i_p} v^{i_1 i_2 \ldots i_p} = 1$.) Moreover, we have

Proposition 1 *We have the following relations*

$$\langle\langle t^l_k,\, \det t \rangle\rangle_c = c^p (-1)^{p-1} q p_q M^l_k, \quad \langle\langle \det t,\, t^l_k \rangle\rangle_c = c^p (-1)^{p-1} q p_q N^l_k, \tag{3}$$

where $M^l_k = u_{i_1 i_2 \ldots i_{p-1} k} v^{l i_1 i_2 \ldots i_{p-1}}$, $N^l_k = u_{k i_1 i_2 \ldots i_{p-1}} v^{i_1 i_2 \ldots i_{p-1} l}$ and $p_q = 1 + q + \ldots + q^{p-1}$. (Let us note that the operators $M = (M^l_k)$ and $N = (N^l_k)$ have been introduced in [G], p. 816.)

Proof. By axiom (i) we have

$$\langle\langle t_k^l, \det t\rangle\rangle_c = c^p u_{i_1 i_2 \ldots i_p} v^{j_1 j_2 \ldots j_p} \langle\langle t_k^{m_1}, t_{j_p}^{i_p}\rangle\rangle\langle\langle t_{m_1}^{m_2}, t_{j_{p-1}}^{i_{p-1}}\rangle\rangle \ldots$$

$$\langle\langle t_{m_{p-1}}^l, t_{j_1}^{i_1}\rangle\rangle = c^p u_{i_1 i_2 \ldots i_p} v^{j_1 j_2 \ldots j_p} S_{j_p k}^{m_1 i_p} S_{j_{p-1} m_1}^{m_2 i_{p-1}} \ldots S_{j_1 m_{p-1}}^{l i_1}.$$

The term $u_{i_1 i_2 \ldots i_p} v^{j_1 j_2 \ldots j_p} S_{j_p k}^{m_1 i_p} S_{j_{p-1} m_1}^{m_2 i_{p-1}} \ldots S_{j_1 m_{p-1}}^{l i_1}$ was found in [G] while commuting the elements $V = v^{j_1 j_2 \ldots j_p} x_{j_1} x_{j_2} \ldots x_{j_p}$ and x_k (cf. Proposition 5.7 from [G]) and is equal to $(-1)^{p-1} q p_q M_k^l$. This proves the first equality. The second one can be proved in the same way using the commutation law of the elements x_k and $V = v^{j_1 j_2 \ldots j_p} x_{j_1} x_{j_2} \ldots x_{j_p}$. $\square$

Corollary 1 *Equations (2) can be satisfied for some $c \in K$ iff the operators M and N are scalar (this property is equivalent by virtue of Proposition 5.9 from [G] to the quantum determinant being central) and, moreover, $M = N$. More precisely, if $M = m \operatorname{id}$, $N = n \operatorname{id}$, $m, n \in K$, and $m = n$ we can satisfy the relations*

$$\langle\langle t_k^l, \det t\rangle\rangle_c = \delta_k^l = \langle\langle\det t, t_k^l\rangle\rangle_c$$

by putting $c^p = (-1)^{p-1} q^{-1} p_q^{-1} m^{-1}$.

Let us note that the operators M and N satisfy the relation $MN = q^{p-1} p_q^{-2} \operatorname{id}$ (cf. [G]). Thus, if $M = m \operatorname{id}$, $N = n \operatorname{id}$, the relation $M = N$ is equivalent to

$$m^2 = q^{p-1} p_q^{-2}. \tag{4}$$

Thus, we have reduced the problem of describing the quantum cogroups $\operatorname{Fun}(SL(S))$ allowing a dual quasitriangular structure to the classification problem of all even Hecke symmetries such that the corresponding operator M is scalar, $M = m \operatorname{id}$, with m satisfying (4). In the next section we will consider this problem for Hecke symmetries of TL type.

Remark 2 Let us observe that if the operators M and N are not scalar, one cannot define the algebra $\operatorname{Fun}(SL(S))$, but it is possible to define the algebra $\operatorname{Fun}(GL(S))$ by introducing a new generator $\det^{-1}$ satisfying the relations $\det^{-1} \det t = 1$, and the commutation law of $\det^{-1}$ with other generators arising from this relation (cf. [G]). Moreover, it is possible to extend the canonical pairing up to that defined on $\operatorname{Fun}(GL(S))$ by setting

$$\langle\langle t_i^p, \det^{-1}\rangle\rangle\langle\langle t_p^j, \det t\rangle\rangle = \delta_i^j, \ \langle\langle\det^{-1}, t_i^p\rangle\rangle\langle\langle\det t, t_p^j\rangle\rangle = \delta_i^j.$$

The details are left to the reader.

Remark 3 Let us observe that if an even Hecke symmetry is of TL type and M is scalar, then $M = N$ since in this case $M = VU$ and $N = UV$ (cf. Section 4). Therefore, if the algebra $\mathrm{Fun}(SL(S))$ is well defined (i.e., the corresponding quantum determinant is central) it automatically has a canonical dual quasitriangular structure. It is not clear whether there exists a Hecke symmetry of rank $p > 2$ such that the algebra $\mathrm{Fun}(SL(S))$ is well defined (i.e., $M = m\,\mathrm{id}$) but the factor m does not satisfy the relation (4) and therefore the corresponding canonical pairing is not compatible with the equation $\det t = 1$.

4   The TL algebra case

Now let us consider the case related to TL algebras. In this case it is possible to give an exhausting classification of the corresponding Hecke symmetries.

Indeed, it is easy to see (cf. [G]) that any even Hecke symmetry of TL type can be expressed by means of the Levi–Civita tensors $U = (u_{ij})$ and $V = (v^{kl})$ in the following way

$$S_{ij}^{kl} = q\delta_i^k\delta_j^l - (1+q)u_{ij}v^{kl}.$$

Then the QYBE and the Hecke second degree relation are equivalent to the system

$$\mathrm{tr}\,UV^t = 1, \;\; UVU^tV^t = q(1+q)^{-2}\,\mathrm{id}. \tag{5}$$

Hereafter $U \to U^t$ is the transposition operator. Thus, $\mathrm{tr}\,UV^t = u_{ij}v^{ij}$.

Introducing the matrix $Z = (1+q)VU^t$ $(z_i^j = (1+q)v^{jk}u_{ik})$ and using the fact that the second relation of (5) can be represented in form $V^tU^tVU = q(1+q)^{-2}\,\mathrm{id}$, we can reduce the relations (5) to the form

$$(Z^t)^{-1}q = V^{-1}ZV, \;\; \mathrm{tr}\,Z = 1+q. \tag{6}$$

The family of all solutions to the QYBE over the field $K = \mathbf{C}$ is described by the following

Proposition 2 [G] *The pair (Z, V) is a solution of the system* (6) *iff the matrix Z is such that $\mathrm{tr}\,Z = 1 + q$ and its Jordan form contains along with any cell corresponding to an eigenvalue x, an analogous cell with eigenvalue q/x (with the same multiplicity).*

Remark 4 Let us note that U and V are transformed under changes of base as bilinear form matrices, while Z is transformed as an operator matrix (their transformations are coordinated and the relations (6) are stable). So, assuming $K = \mathbf{C}$, we can represent the operator Z in Jordan form by an appropriate choice of base. Moreover, we can assume that the cells with eigenvalues x and q/x are in positions symmetric to each other with respect to the center of the matrix Z. Observe that if the number of the cells is odd the eigenvalue of the middle one is $\pm\sqrt{q}$.

It is not difficult to see that for such a choice of base the tensor V can be taken in the form of a skew-diagonal matrix (i.e., possessing nontrivial terms only at the auxiliary diagonal). Let us fix such a matrix V_0 and note that all other V satisfying (6) are of the form $V = WV_0$, where W commutes with Z. In the sequel we assume that a base possessing these properties is fixed.

Let us observe that in case under consideration we have $M = UV$, $N = VU$. Moreover, relations (4) take the form $m^2 = q(1+q)^{-2}$. Using the relation $U^t = (1+q)^{-1} V^{-1} Z$, we can transform the equality $UV = m\,\mathrm{id}$ to

$$Z = (1+q)mV(V^t)^{-1}. \tag{7}$$

Let us assume that Z has a simple spectrum, i.e., its eigenvalues are pairwise distinct. So, its Jordan form is diagonal: $Z = \mathrm{diag}(z_1, \ldots, z_n)$. The family of diagonal Z, satisfying conditions of Proposition 2 and fulfilling the only relation $\mathrm{tr}\, Z = 1 + q$, can be parametrized by $(z_1, \ldots, z_r)$ with $r = n/2$, if n is even, and with $r = (n-1)/2$, if n is odd (if n is odd we have also a choice for the value of $z_{r+1} = \pm\sqrt{q}$).

Since Z has a simple spectrum, any W commuting with Z is also diagonal (with arbitrary diagonal entries). This implies that V satisfies (6) iff it is skew-diagonal with arbitrary entries at the auxiliary diagonal. Therefore U is also skew-symmetric. Thus, we have $v^{ij} \neq 0$, $u_{ij} \neq 0$ iff $i+j = n$. For the sake of simplicity, we will use the notation v^i (u_i) instead of $v^{i\,n+1-i}$ ($u_{i\,n+1-i}$). Let us note that $z_i = (1+q)u_i v^i$ (up to the end of this section there is no summation over repeated indices).

It is easy to see that relation (7) is satisfied iff the entries v_i fulfill the system

$$m(1+q)v^i/v^{n-i+1} = z_i, \ 1 \leq i \leq n. \tag{8}$$

This system is consistent by virtue of the relations

$$z_i\, z_{n-i+1} = q, \ m^2 = q(1+q)^{-2}.$$

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Moreover, the family of the solutions of the system (8) can be parametrized by $(v_1, \ldots, v_r)$. Let us note that if n is odd, the value of $z_{r+1} = \pm\sqrt{q}$ depends on that of $m = \pm\sqrt{q}(1+q)^{-1}$, namely, we have $z_{r+1} = m(1+q)$.

Thus, we have proved the following

Proposition 3 *Let $K = \mathbf{C}$, S be a Hecke symmetry of TL type and Z be the corresponding tensor described in Proposition 2 with a simple spectrum (a parametrization of all such tensors Z was given above). Then the dual quasitriangular structure defined on the algebra $\mathbf{A}(S)$ can be descended on the quantum cogroup $\mathrm{Fun}(SL(S))$ iff V is a skew-diagonal matrix with the entries $v_{i\,n+1-i} = v_i$ satisfying the system (8). This system is always compatible and the family of its solutions can be parametrized as above.*

Remark 5 Let us observe that the Hecke symmetries of TL type such that the operator UV is scalar are just those introduced in [DL] (the authors of [DL] use another normalization of the operator S).

5 Nondegeneracy of the canonical pairing

In the present section we show that when $n = \dim \mathbf{V} > 2$ and q is generic, the canonical pairing $\langle\langle \, , \, \rangle\rangle_c$ is nondegenerate for those even Hecke symmetries of TL type whose operator Z has a generic simple spectrum. As above, we assume that Z has a diagonal form in a chosen base and therefore the tensors U and V are skew-diagonal. In the sequel we put $c = 1$.

Thus we have the pairing $\langle\langle t_i^j, t_k^l \rangle\rangle = S_{ki}^{jl}$. To show that it is nondegenerate we will compute the Gram determinant, i.e., the determinant of the Gram matrix. The rows and the columns of this matrix are labeled by the bi-index (i, j) running over the set

$$(1, 1), \ldots, (1, n), (2, 1), \ldots, (2, n), \ldots, (n, 1), \ldots, (n, n).$$

So, the term $\langle\langle t_i^j, t_k^l \rangle\rangle = R_{ik}^{jl} = S_{ki}^{jl}$ is situated at the intersection of the (i, j)-row and the (k, l)-column.

Let us note that if S is a Hecke symmetry of TL type, then all the entries of the matrix S_{ki}^{jl} are equal to zero unless either $i = j$, $k = l$ or $i + k = j + l = n + 1$. So, we have just two nonzero elements in the (i, j)-row

namely, R_{ij}^{ji} and R_{in+1-i}^{jn+1-j}, if $i+j \neq n+1$, and only one, namely, R_{in+1-i}^{n+1-ii}, if $i+j = n+1$. A similar statement is valid for the columns.

This yields that if $i+j \neq n+1$ then the (i,j)- and $(n+1-j, n+1-i)$-rows and (j,i)- and the $(n+1-i, n+1-j)$-columns possess just four nontrivial elements

$$R_{ij}^{ji}, \ R_{in+1-i}^{jn+1-j}, \ R_{n+1-jj}^{n+1-ii}, \ R_{n+1-jn+1-i}^{n+1-in+1-j}$$

situated at their intersections. If $i+j = n+1$, then two rows (columns) are merged into one and the only nontrivial element R_{in+1-i}^{n+1-ii} belongs to the intersection of the $(i, n+1-i)$-row and the $(n+1-i, i)$-column.

For example, for $n=3$ we have the following Gram matrix

$$\mathbf{G} = \begin{pmatrix} R_{11}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{13}^{13} \\ 0 & 0 & 0 & R_{12}^{21} & 0 & 0 & 0 & R_{13}^{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R_{13}^{31} & 0 & 0 \\ 0 & R_{21}^{12} & 0 & 0 & 0 & R_{22}^{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_{22}^{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{22}^{31} & 0 & 0 & 0 & R_{23}^{32} & 0 \\ 0 & 0 & R_{31}^{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_{31}^{22} & 0 & 0 & 0 & R_{32}^{23} & 0 & 0 & 0 \\ R_{31}^{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{33}^{33} \end{pmatrix}.$$

By changing the order of the rows and columns, we can reduce this matrix to a block-diagonal form, where all blocks are either one-dimensional and consist of elements R_{in+1-i}^{n+1-ii} or two-dimensional and have the following form

$$\begin{pmatrix} R_{ij}^{ji} & R_{in+1-i}^{jn+1-j} \\ R_{n+1-jj}^{n+1-ii} & R_{n+1-jn+1-i}^{n+1-in+1-j} \end{pmatrix}.$$

Denote by $I(n)$ the set of indices i,j satisfying $1 \leq i, j \leq n, i+j \neq n+1$. Then $\det \mathbf{G}$ is equal up to a sign to

$$\prod_{1 \leq i \leq n} R_{in+1-i}^{n+1-ii} \sqrt{\left| \prod_{I(n)} \det \begin{pmatrix} R_{ij}^{ji} & R_{in+1-i}^{jn+1-j} \\ R_{n+1-jj}^{n+1-ii} & R_{n+1-jn+1-i}^{n+1-in+1-j} \end{pmatrix} \right|}.$$

(The root is motivated by the fact that any factor in the second product is taken two times.)

By straightforward calculations it is not difficult to see that

$$\prod_{1\le i\le n} R_{i\,n+1-i}^{n+1-i\,i} = \prod_{1\le i\le n}(q-z_i)$$

and

$$\prod_{I(n)} \det \begin{pmatrix} R_{ij}^{ji} & R_{i\,n+1-i}^{j\,n+1-j} \\ R_{n+1-j\,j}^{n+1-i\,i} & R_{n+1-j\,n+1-i}^{n+1-i\,n+1-j} \end{pmatrix} = \prod_{I(n)}(q^2 - z_{n+1-i}z_j).$$

Thus, we have proven the following

Proposition 4

$$(\det \mathbf{G})^2 = \prod_{1\le i\le n}(q-z_i)^2 \prod_{I(n)}(q^2 - z_{n+1-i}z_j).$$

It is interesting to observe that the final expression depends only on the matrix Z. This enables us to state the following

Proposition 5 *Let us assume that $S : \mathbf{V}^{\otimes 2} \to \mathbf{V}^{\otimes 2}$ is an even Hecke symmetry of TL type, the corresponding operator Z possesses a simple spectrum and $n = \dim \mathbf{V} \ge 4$. Then for a generic q and for a generic Z (of such type) the canonical pairing is nondegenerate.*

Proof. The set where the determinant $\det \mathbf{G}$ vanishes is an algebraic variety in the space $\mathbf{C}^{n+1}$ generated by the indeterminates z_i and q. It suffices to show that this variety is not contained in that defined by

$$z_i\, z_{n+1-i} = q, \quad \sum z_i = 1 + q.$$

Let us decompose the expression

$$\prod_{I(n)}(q^2 - z_{n+1-i}z_j)$$

into the product of factors with $i = j$ and those with $i \ne j$.

If $i = j$, we have $z_{n+1-i}z_j = q$. Thus, the above product is equal to

$$(q^2 - q)^n \prod_{I(n),\, i\ne j}(q^2 - z_{n+1-i}z_j).$$

So, for q such that $q \ne 0, q \ne 1$, we have $\det \mathbf{G} = 0$ iff $z_i = q$ for some i or $q^2 = z_{n+1-i}z_j$ for some $i \ne j$.

From the above parametrization it is evident that if $n \geq 4$ there exists a matrix $Z = \mathrm{diag}(z_1, \ldots, z_n)$ satisfying the conditions of Proposition 2 and such that $\det \mathbf{G} \neq 0$.

Let us consider the case $n = 3$ separately. In this case the set of such diagonal matrices Z is parametrized by z_1 (after choosing a value of $z_2 = \pm\sqrt{q}$) satisfying the equation $z_1 \pm \sqrt{q} + q/z_1 = 1 + q$. This equation has two solutions for any choice of the value $\pm\sqrt{q}$. It is not difficult to see that for a generic q we have $\det \mathbf{G} \neq 0$ for any of these four values of z_1. $\square$

Let us note that the canonical pairing becomes degenerate if $q = 1$ (this case corresponds to an involutive symmetry ($S^2 = \mathrm{id}$)) or if $n = 2$, since in this case the system $z_1 z_2 = q$, $z_1 + z_2 = 1 + q$ has two solutions $z_1 = 1$, $z_2 = q$ and $z_1 = q$, $z_2 = 1$ for which the product $(q - z_1)(q - z_2)$ vanishes. And we always have $\det \mathbf{G} = 0$.

This is the principal reason why the latter case ($n = 2$) which corresponds to a quasiclassical Yang–Baxter operator S (it is in fact the only quasiclassical case related to the TL algebra) differs crucially from the nonquasiclassical ones ($n > 2$).

Thus, according to the above construction, we can convert the right $\mathrm{Fun}(SL(S))$-comodules $\mathbf{V}^{\otimes m}$ into left $\mathrm{Fun}(SL(S))^{op}$-modules (assuming S to be an even Hecke symmetry of TL type, q to be generic and Z to have a simple spectrum, also generic) and therefore into right $\mathrm{Fun}(SL(S))$-modules.

6 Discussion of a possible representation theory

Our further aim is to construct some representation theory of the algebra $\mathrm{Fun}(SL(S))$ equipped with the above action. Conjecturally, it looks like that of $SL(2)$. Let us denote V_m the symmetric component of $\mathbf{V}^{\otimes m}$. (Let us note that in classical and quasiclassical cases $m/2$ is just the spin of the representation $U(sl(2)) \to \mathrm{End}(V_m)$ or $U_q(sl(2)) \to \mathrm{End}(V_m)$.)

It seems very plausible that similarly to the $SL(2)$- or $U_q(sl(2))$-case we have for a generic q the following properties

- the $\mathrm{Fun}(SL(S))$-modules V_m are irreducible,

- any irreducible finite-dimensional $\mathrm{Fun}(SL(S))$-module is isomorphic to one of V_m,

- any finite-dimensional $\mathrm{Fun}(SL(S))$-module is completely reducible,

- we have the classical formula $V_i \otimes V_j = \oplus_{|i-j| \le k \le i+j} V_k$.

To motivate the latter formula, let us show that it is satisfied at least "in sense of dimensions", i.e.,

$$\dim V_i \otimes \dim V_j = \sum_{|i-j| \le k \le i+j} \dim V_k. \tag{9}$$

Indeed, using the fact that the Poincaré series of the symmetric algebra of the space $\mathbf{V}$ is equal to $(t^2 - nt + 1)^{-1}$, one can see that

$$\dim V_i = \alpha^i + \alpha^{i-2} + \alpha^{i-4} + \ldots + \alpha^{-i},$$

where $\alpha = n/2 + \sqrt{(n/2)^2 - 1}$ is a root of the equation $t^2 - nt + 1 = 0$. Then relation (9) can be established by straightforward calculations. The details are left to the reader.

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